REVIEWS OF MODERN PHYSICS VOLUME 19, NUMBER 4 OCTOBER, 1947

The Relativity Effect in Planetary Motions

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INTRODUCTION

IT is well known that, according to the general theory of relativity,[1] the elliptical orbit of a planet referred to a Newtonian frame of reference rotates in its own plane in the same direction as the planet moves, with a speed that is given by

$$\frac{\delta\tilde{\omega}}{\varphi}=\frac{12\pi^2a^2}{c^2T^2(1-e^2)}.$$

In this formula $\delta\tilde{\omega}/\varphi$ is the amount of rotation (commonly called the motion of the perihelion) per revolution of the planet about the sun, a is half the major axis of the ellipse, c is the velocity of light, T is the time required for one revolution of the planet, and e is the eccentricity of the ellipse; if a, c, and T are measured in centimeters and seconds. The fraction of a revolution through which the perihelion advances during one revolution of the planet is represented by $\delta\tilde{\omega}/\varphi$, a dimensionless number. For comparison with observations it is convenient to express $\delta\tilde{\omega}/\varphi$ in seconds of arc per century. Table I gives the theoretical effects derived from the formula for the five inner planets under the name of $\tilde{\omega}'$, based on a value of the solar parallax of 8″.790. The last column gives for each planet the motion of the perihelion multiplied by the eccentricity of the orbit; the size of this quantity is a measure of the angular displacement of the planet when it is at perihelion, and hence this is the quantity that fixes the accuracy with which the effect can be determined by analysis of observations.

It is at once evident that the effect can be detected most easily in the motion of Mercury. Indeed, Einstein's announcement of the general theory of relativity in its definitive form[2] was immediately hailed by some astronomers as explaining a previously unaccountable discrepancy between the observed and theoretical motions of this planet. Others were, however, intuitively opposed to relativity, and they directed attention to a small discrepancy yet remaining as evidence that the theory of relativity could not be correct; the relativists contended that the small remaining discrepancy was due to errors either in the observations or in the classical theory of the motion. In justice it should be said that the questions involved are not simple ones, but are complicated by three causes: (1) Observations of Mercury are among the most difficult in positional astronomy. They have to be made in the daytime, near noon, under unfavorable conditions of the atmosphere; and they are subject to large systematic and accidental errors arising both from this cause and from the shape of the visible disk of the planet. (2) The planet's path in Newtonian space is not an ellipse but an exceedingly complicated space-curve due to the disturbing effects of all of the other planets. The calculation of this curve is a difficult and laborious task, and significantly different results have been obtained by different computers. (3) The observations cannot be made in the Newtonian frame of reference. They are referred to the moving equinox, that is, they are affected by the precession of the equinoxes, and the determination of the precessional motion is one of the most difficult problems of positional astronomy, if not the most difficult. In the light of all these hazards it is not surprising that a difference of opinion could exist regarding the closeness of agreement between the observed and theoretical motions.

I am not aware that relativity is at present regarded by physicists as a theory that may be believed or not, at will. Nevertheless, it may be of some interest to present the most recent evidence on the degree of agreement between the observed and theoretical motions of the planets, which is the object of this article. The evidence is of two kinds: (1) a discussion by the author[3]

[1] See, e.g., A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Teddington, England, 1924), second edition, p. 89.

[2] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," Ann. d. Physik **49**, 769 (1916).

[3] G. M. Clemence, "The motion of Mercury 1765–1937," Astro. Pap. Am. Ephemeris **11**, 1 (1943).

of observations of Mercury from 1765 to 1937 which was intended to exhaust the useful observational evidence available on the motion of Mercury at that time. In this discussion the relativity effect in the motion of Mercury is confirmed, and some slight evidence of the effect is found in the motion of the earth as well. (2) a discussion by H. R. Morgan[4] of observations of the sun, in which he concludes that the effect is present in the motion of the earth.

In order to render the subject more readily comprehensible, the results are presented here under a different form from that heretofore published and the numerical values have been altered slightly in three ways. Doolittle's calculation[5] of the Newtonian motions, with certain corrections, is used instead of Newcomb's, some new values of the planetary masses are introduced, and Oort's most recent value[6] of the precession is adopted. The observational results remain unchanged. It may be remarked that the effect of the alterations has been to make the agreement between observations and theory slightly worse instead of better, but not significantly so.

THE OBSERVED MOTIONS OF THE PERIHELIA OF MERCURY AND THE EARTH

Unfortunately, the observational material is so extensive and the methods of analysis so complex that it is not practicable here to present any evidence that will enable the reader to form an independent judgment of the errors involved. All that can be done is to give a very brief description of the methods employed, and the numerical results. This is not to minimize the importance of the error estimates, which are, of course, the most critical feature of the entire work; the interested reader will, it is hoped, find a sufficient discussion of the errors in the references.

The term "probable error," whenever it is used in what follows, is not to be understood as meaning the quartile error obtained by multiplying the standard deviation by 0.6745. It is well known that the quartile error measures only the accidental discordances of a set of data, no allowance being made for systematic errors, which in an analysis of a very extended series of observations are likely to be much more important than the accidental discordances. The probable errors given here are in every instance larger than the quartile errors, and they correspond more nearly to what Dorsey[7] has called the "dubiety." I have obtained them by adding to the quartile errors the quantity which, as nearly as I could judge, represents the size of the largest systematic error that could affect the results. It is difficult to define in precise language a quantity that depends on a multiplicity of personal judgments, nevertheless the attempt must be made. By probable error I mean that quantity which, when added to and subtracted from a result, gives a range within which the probability for the inclusion of the true value is one-half; more precisely, the probable error is intended to measure the discordance of all future determinations in addition to those in the past.

TABLE I. Theoretical values of the advance of the perihelia per century.

Planet	$\tilde{\omega}'$	$e\tilde{\omega}'$
Mercury	43″.03	8″.847
Venus	8.63	0.059
Earth	3.84	0.064
Mars	1.35	0.126
Jupiter	0.06	0.003

The observations of Mercury are of two different kinds: observations of its spherical coordinates on the celestial sphere when it is on the meridian, and observations of the time at which its disk is tangent to the disk of the sun when Mercury crosses the face of the sun. The meridian observations extend from 1765 to 1937 and number about 10,000 in each coordinate. Observations of 17 transits have been used, extending from 1799 to 1940.

The observed coordinates are not discussed directly, but instead the small differences between the observed coordinates and those calculated from a theory of the motions are used.

[4] H. R. Morgan, "The earth's perihelion motion," Astro. J. **50**, 127 (1945).

[5] Eric Doolittle, "The secular variations of the elements of the orbits of the four inner planets computed for the epoch 1850.0 G.M.T.," Trans. Am. Phil. Soc. **22**, 37 (1925).

[6] J. H. Oort, "The constants of precession and of galactic rotation," Bull. Astro. Inst. Netherlands **9**, 424 (1943).

[7] N. E. Dorsey, "The velocity of light," Trans. Am. Phil. Soc. **34**, 1 (1944).

Each of these differences gives rise to an equation of condition, the unknown quantities being corrections to the constants used in the calculated coordinates. These equations are collected into groups extending over about ten years each and solved by the method of least squares. The number of unknown quantities is twelve, one of them being the correction to the assumed, or tabular, position of the perihelion. In principle, a number of corrections at successive epochs to this assumed position of the perihelion are obtained, and the sum of the corrections gives the correction to the assumed *motion* of the perihelion. The procedure followed with the transits of Mercury is much the same, except that the whole series of transits furnishes only two equations of condition because transits can occur only in two narrow regions of Mercury's orbit. These two additional conditions are imposed on the final results of the meridian observations, and another adjustment is made by least squares.

Since observations of Mercury do not give the absolute position of the planet in space but only the direction of a line from the planet to the observer, they depend equally on the position of Mercury and the position of the earth, and the motion of the earth's perihelion may be introduced also as an unknown to be determined. Determination of motion made in this way is for several reasons inferior in accuracy to that obtained from observations of the sun, but the determination has some value.

The observations of the sun are more numerous than are those of Mercury, but they extend over about the same length of time. The analysis is simpler because fewer unknowns have to be determined, but the principles involved are the same.

For the total observed rate of motion of Mercury's perihelion at 1850, referred to the moving equinox, I have found, in seconds of arc per Julian century of 36,525 mean solar days, 5599.74±0.41. For the earth I have found from observations of Mercury 6182.0±3.6, and Morgan has obtained from observations of the sun 6183.9±1.2. Weighting the last two determinations in accordance with their assigned probable errors gives, as the definitive result to be used here for the observed motion of the earth's perihelion, 6183.7±1.1.

THE THEORETICAL MOTIONS OF THE PERIHELIA

The theoretical motions of the perihelia, referred to the moving equinox, are obtained by adding together the parts contributed by the gravitational actions of the several planets (and in the case of the earth the portion arising from the non-sphericity of the earth-moon system), the rotational oblateness of the sun, the general precession in longitude, and the relativity effect. The separate contributions are shown in Table II,

TABLE II. Contributions to the motion of the perihelia of Mercury and the earth.

Cause	m^{-1}	Motion of perihelion: Mercury	Motion of perihelion: Earth
Mercury	6 000 000 ±1 000 000	0″.025±0″.00	−13″.75±2″.3
Venus	408 000 ± 1 000	277.856±0.68	345.49±0.8
Earth	329 390 ± 300	90.038±0.08	
Mars	3 088 000 ± 3 000	2.536±0.00	97.69±0.1
Jupiter	1 047.39± 0.03	153.584±0.00	696.85±0.0
Saturn	3 499 ± 4	7.302±0.01	18.74±0.0
Uranus	22 800 ± 300	0.141±0.00	0.57±0.0
Neptune	19 500 ± 300	0.042±0.00	0.18±0.0
Solar oblateness		0.010±0.02	0.00±0.0
Moon			7.68±0.0
General precession (Julian century, 1850)		5025.645±0.50	5025.65±0.5
Sum		5557.18 ±0.85	6179.1 ±2.5
Observed motion		5599.74 ±0.41	6183.7 ±1.1
Difference		42.56 ±0.94	4.6 ±2.7
Relativity effect		43.03 ±0.03	3.8 ±0.0

and for convenience the relativity effect is omitted from the upper part of the table. The discrepancy between the observed motion and the incomplete theory may then be compared directly with the relativity effect, which is given on the last line of the table.

The contributions of the planets are directly proportional to their several masses, which are not all known with the desired accuracy. The quantities denoted by m^{-1} are the reciprocals of the adopted masses, the sun's mass being taken as unity, and the attached probable errors give rise to the probable errors associated with the theoretical contributions to the motions. In the case of Mercury each planetary contribution (except that of Mercury itself) is the sum of three parts: the motion of the perihelion in the plane of the orbit, the contribution arising from the motion of the node, and the contribution from the motion of the ecliptic. These last two effects arise from the way in which the longitude of the perihelion is measured; from the equinox along the ecliptic to the node, and then along the orbit of Mercury to the perihelion. The figures given depend on the calculations of Doolittle,[5] but his values of the masses have been altered.

The probable errors of the masses are my own estimates. It is evident from Table II that the uncertainties in the masses of Mercury and Venus contribute most to the uncertainty of the final results; indeed, until these two masses are better determined, the motions of the perihelia of Mercury and the earth can be observed more accurately than they can be calculated. A thorough discussion of meridian observations of Venus would probably give a value of the mass of Mercury with a probable error about fifty percent of that given here. Increased accuracy in the mass of Venus must await the completion of the theory of the motion of Mars now in progress. The most uncertain of the probable errors is that attached to Mercury; de Sitter[8] in 1938 estimated it to be fifty percent larger than the value given here. The estimation of this probable error is a very difficult matter and it may be that de Sitter's guess is better than mine; in this case the penultimate line of the table would read $4''.6\pm3''.7$ instead of $4''.6\pm2''.7$.

The effect of the rotational oblateness of the sun is to produce a small additional contribution to the perihelion motions of the planets. The general theory of such effects has been discussed by Brouwer.[9] It is known[10] that if the sun were a homogeneous gas sphere the resulting contribution to the centennial motion of Mercury's perihelion would be $1''.2$. For the actual sun this value must be multiplied by $4K/3$, K being a dimensionless constant depending on the interior constitution. The value of K is very small for a highly concentrated gas sphere, which the sun is believed to be; Russell[11] has given empirical values deduced from the observed motions of double stars, the more reliable of which range up to 0.02. The latest theoretical determination is that of Motz,[12] who finds 0.006. I adopt the latter value with a probable error of twice its amount, which gives for the centennial perihelion motion of Mercury $0''.010\pm0''.02$; the probable error is very uncertain. The effect on the earth is much smaller than for Mercury.

The precession is that resulting from Oort's latest discussion;[6] the attached probable error is my estimate.

The probable errors attached to the theoretical relativity-effects correspond to a probable error in the solar parallax of $\pm0''.003$.

CONCLUSION

The theoretical relativity effect in the motion of Mercury's perihelion is $43''.03\pm0''.03$; the value obtained by subtracting all other known effects from the total observed motion is $42''.56\pm0''.94$. For the earth's perihelion the corresponding figures are $3''.8\pm0''.0$ and $4''.6\pm2''.7$. The confirmation by observation of the relativity effect is regarded as satisfactory for both Mercury and the earth.

As soon as the gravitational theory of Mars is placed on a sound basis, the relativity effect in the motion of this planet should be easily detected with higher precision than has been found for the earth.

I am indebted to Professor Schilt and Professor Schwarzschild of Columbia University for valuable aid in connection with this work.

[8] W. de Sitter, edited and completed by Dirk Brouwer, "On the system of astronomical constants," Bull. Astro. Inst. Netherlands **8**, 213 (1938).

[9] Dirk Brouwer, "The motion of a particle with negligible mass under the gravitational attraction of a spheroid," Astro. J. **51**, 223 (1945).

[10] F. Tisserand, Traité de Mécanique Céleste **4**, 537 (1896).

[11] H. N. Russell, Note on ellipticity in eclipsing binaries, Astrophys. J. **90**, 641 (1939).

[12] Lloyd Motz, "The apsidal motion in binary stars built on a point-source convective-core model with varying guillotine factor," Astrophys. J. **94**, 253 (1941).